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OPEN

DOI 10.1515/tmj-2017-0039

Notes on summability in $QTAG$-modules

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Abstract

We show the inheritance of summable property for certain fully invariant submodules by the $QTAG$-modules and vice versa. Important generalizations and extensions of classical results in this direction are also established.

2010 Mathematics Subject Classification. **16K20**
Keywords. summable modules, fully invariant submodules, α-large submodules.

1 Introduction, notation and other conventions

A right module M over an associative ring with unity is a $QTAG$-module if every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules. This is a very fascinating structure that has been the subject of research of many authors. Different notions and structures of $QTAG$-modules have been studied, and a theory was developed, introducing several notions, interesting properties, and different characterizations of submodules. Many interesting results have been obtained, but there is still a lot to explore.

All rings below are assumed to be associative and with nonzero identity element; all modules are assumed to be unital $QTAG$-modules. A module in which the lattice of its submodule is totally ordered is called a serial module; in addition if it has finite composition length it is called a uniserial module. Let us recall some definitions from [18, 19]. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module and for any R-module M with a unique decomposition series, $d(M)$ denotes its decomposition length. For a uniform element $x \in M$, $e(x) = d(xR)$ and $H_M(x) = \sup \left\{ d\left(\frac{yR}{xR}\right) : y \in M,\ x \in yR \text{ and } y \text{ uniform} \right\}$ are the exponent and height of x in M, respectively. $H_n(M)$ denotes the submodule of M generated by the elements of height at least n and $H^n(M)$ is the submodule of M generated by the elements of exponents at most n [14]. Let us denote by M^1, the submodule of M, containing elements of infinite height. As defined in [16], the module M is h-divisible if $M = M^1 = \bigcap_{n=0}^{\infty} H_n(M)$. The module M is h-reduced if it does not contain any h-divisible submodule. In other words it is free from the elements of infinite height. The module M is called separable [13] if $M^1 = 0$. A submodule N of M is said to be high [15], if it is a complement of M^1 i.e., $M = N \oplus M^1$. For an ordinal α, a submodule $N \subseteq M$ is an α-high submodule [5] of M if N is maximal among the submodules of M that intersect $H_\alpha(M)$ trivially. An ordinal α is said to be confinal with ω, if α is the limit of a countable ascending sequence of ordinals [4].

Tbilisi Mathematical Journal 10(2) (2017), pp. 235–242.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 08 November 2016.
Accepted for publication: 05 May 2017.

For an ordinal σ, a submodule N of M is said to be σ-pure, if $H_\beta(M) \cap N = H_\beta(N)$ for all $\beta \leq \sigma$ and a submodule N of M is said to be isotype in M, if it is σ-pure for every ordinal σ [9]. For a submodule $N \subseteq M$, the valuation of N induced by height in M is defined by $v(x) = H_M(x)$, the height of x in M, for all $x \in N$ and $N = K \oplus L$ is a valuated direct sum if $v(k+\ell) = \min\{v(k), v(\ell)\}$ for all $k \in K$ and $\ell \in L$ [2].

A family $\mathcal{N}$ of nice submodules of M is called a nice system [6] in M if the following hold:
(i) $0 \in \mathcal{N}$;
(ii) if $\{N_i\}_{i\in I}$ is any subset of $\mathcal{N}$, then $\sum\limits_{i\in I} N_i \in \mathcal{N}$;
(iii) given any $N \in \mathcal{N}$ and any countable subset X of M, there exists $K \in \mathcal{N}$ containing $N \cup X$, such that K/N is countably generated.

It is interesting to note that almost all the results which hold for TAG-modules are also valid for $QTAG$-modules [9]. Our notations and terminology are standard and may be found in the texts [10, 11, 12].

An h-reduced $QTAG$-module M is totally projective if it has a nice system, and direct sums and direct summands of totally projective modules are also totally projective [3]. There are numerous characterizations of totally projective modules more enlightening than the definition, but the main fact that we need is the following observation: If M is totally projective, then so are $H_\beta(M)$ and $M/H_\beta(M)$ for all ordinals β; and, conversely, if there is an ordinal β such that both $H_\beta(M)$ and $M/H_\beta(M)$ are totally projective, then M is totally projective. (Note that, in particular, the same claim follows for direct sums of countably generated modules).

A problem of interest is to find suitable properties of modules that are inherited by special submodules called fully invariant. Recall that a submodule F of a module M is said to be fully invariant [8] if each endomorphism of M sends F into itself. In [1], the first author proved that if F is a fully invariant submodule of the totally projective module M, then both F and M/F are totally projective modules; and, conversely, if both F and M/F are totally projective, then M is itself totally projective, where the fully invariant submodule F of M is of concrete type. The general case for F is still unanswered.

The purpose that motivates the writing of the present article is to try to modify to what extent the quoted above results for totally projective modules can be proved for an independent class of so-called summabble modules, thus generalizing all of the aforementioned attainments. We also concern some partial situations of the general case for the fully invariant submodule F of M; it is noteworthy that $H_\beta(M)$ is a special fully invariant submodule of M.

2 Chief results

For facilitating of the exposition and for the convenience of the readers, we recall the following definition, see ([17]).

Definition 2.1. An h-reduced $QTAG$-module M is summable if $Soc(M) = \oplus_{\beta<\alpha} S_\beta$, where S_β is the set of all elements of $H_\beta(M)$ which are not in $H_{\beta+1}(M)$, where α is the length of M.

To develop the study, we need to prove some results and we start with the following theorem.

Theorem 2.2. Let M be a $QTAG$-module such that $H_\beta(M)$ and $M/H_\beta(M)$ are summable for some ordinal β. Then M is summable, provided length of $M \leq \mu$.

Proof. If $\beta \geq \mu$, we are done. For the remaining case $\beta < \mu$ we write $Soc(M) = Soc(H_\beta(M)) \oplus K_\beta$ where K_β is isometric to the socle of some β-high submodule of M. Assume that N is such a β-high submodule. Since $N \cong (N \oplus H_\beta(M))/H_\beta(M)$ where the latter module is isotype in $M/H_\beta(M)$, a module of countable length, we get that N must be summable. Therefore, $Soc(N)$ must be free, hence so does K_β. But $Soc(H_\beta(M))$ is free too and henceforth a simple technical argument applies to get that $Soc(M)$ is free which gives the desired summability of M. Q.E.D.

Now we give an example to showing that the assertion holds valid even for lengths equal to μ.

Example 2.1. Let $M = N \oplus K$, where N is of countable length whereas K is of length μ a direct sum of countably generated modules. We claim that if $H_\beta(M)$ and $M/H_\beta(M)$ are summable for some $\beta \leq \mu$, then M is summable. Indeed, length of $M = \mu$ and we see that $\beta = \mu$ obviously insures that M is summable. If now $\beta < \mu$, we see that $H_\beta(M) = H_\beta(N) \oplus H_\beta(K)$ and $M/H_\beta(M) \cong (N/H_\beta(N)) \oplus (K/H_\beta(K))$ implying that $H_\beta(N)$ and $N/H_\beta(N)$ are summable as being a direct summands. Thus, by Theorem 2.2, N has to be summable, whence so is M, as claimed.

We continue with other example.

Example 2.2. Let $M = N \oplus K$, where K is of the uncountable length $\omega_1 = \mu$ or when K is not a direct quotient of M. In that direction, whether or not M being summable plus M/K being countable generated with K is isotype in M do imply that K is summable?

Assume that M is countably generated module. Thus we write $M = \cup_{n<\omega} M_n$ and $M_n \subseteq M_{n+1}$. Moreover, all M_n/K are finitely generated and if we assume K is h-pure in M, hence in M_n, consulting with the direct quotient statement, we establish $M_n = K \oplus L_n$ for some finitely generated $L_n \subseteq M_n$ and for every $n < \omega$. So, all M_n are summable, but perhaps M need not be summable provided the length of $M = \omega_1$ or M_n are not isotype in M.

We also emphasize that $Soc(M) \cong Soc(K) \oplus Soc(M/K)$ and

$$Soc(H_\beta(M)) \cong Soc(H_\beta(K)) \oplus Soc((H_\beta(M) + K)/K)$$

for every $\beta \geq 1$.

Now we are ready with an observation on Theorem 2.2.

Example 2.3. The condition on M being h-reduced, can be added. In fact, select a $QTAG$-module M such that $M/H_\omega(M)$ is a direct sum of uniserial modules, $H_\omega(M)$ is h-divisible and $Soc(H_\omega(M))$ is infinite countably generated. Therefore M is a Σ-module but not a summable module. Of course, $H_\omega(M)$ is countably generated h-divisible module and thus, M is a direct sum of a countably generated module and of a direct sum of uniserial modules. On the other hand, N_M (a high submodule of M) is a direct sum of uniserial modules. Because the submodule N_M is h-pure in M, the direct decomposition $Soc(M) = Soc(N_M) \oplus Soc(H_\omega(M))$ guarantees that $Soc(M/N_M) \cong Soc(M)/Soc(N_M) \cong Soc(H_\omega(M))$ is infinite countably generated whence so is the h-divisible M/N_M. Now, bearing in mind that N_M is isotyoe in M, we are done.

We now turn to the question: If F is a fully invariant submodule of M and both F and M/F are summable, does it follow that M is summable as well? (For the corresponding results concerning totally projective modules we refer once again to [1]). In some instance this is so; for examples: (i) $F = Soc^n(M)$ for some natural n. Then $M/F \cong H_n(M)$ is summable only when M is summable. (ii) $F = H_\beta(M)$ for some ordinal number β. Then our forgoing theorem allows us to conclude that the above question holds true in that case.

Now, we will make an attempt to examine the general case. Let $\beta = \{\beta_k\}_{k<\omega}$ be an increasing sequence of ordinals and symbols ∞; that is, for each k, either β_k is an ordinal or $\beta_k = \infty$ and $\beta_k < \beta_{k+1}$ provided $\beta_k \neq \infty$. Imitating [1], with each such sequence β we associate the fully invariant submodule M^β of the $QTAG$-module M as

$$M^\beta = \{x \in M : x \in H_{\beta_k - k}(M) \text{ for all } k < \omega\}.$$

If M is totally projective, then all of its fully invariant submodules are of this form. However, this is an open problem for arbitrary modules. That is why, we specify that all our fully invariant submodules of M in the sequel are taken of the above form presented.

A $QTAG$-module M is an α-module, where α is a limit ordinal, if $M/H_\beta(M)$ is totally projective for every ordinal $\beta < \alpha$. The concept of α-modules were introduced and different results were obtained in terms of α-basic submodules and α-large submodules in [4]. Recall that B is an α-basic submodule of an α-module M if B is totally projective of length at most α, B is α-pure submodule of M, and M/B is h-divisible. Moreover, a fully invariant submodule L of the α-module M is α-large if $M = B + L$, for all α-basic submodules B of M.

We have accumulated all the machinery necessary to prove the following.

Theorem 2.3. Let F be an unbounded, fully invariant submodule of the α-module M, where M has length $\alpha \leq \mu$. Then, F is summable only if M is summable.

Proof. As the text says, we must show that F being summable implies that M is summable. In fact, since $F = M^\beta$, where

$$\beta = (\beta(0), \beta(1), \ldots, \beta(k), \ldots),$$

We set $\gamma = \sup\{\beta(k) : k < \omega\}$. It is not difficult to verify that $H_\omega(F) = H_\gamma(M)$. Hence $H_\gamma(M)$ is summable. If $\gamma < \alpha$, we have that $M/H_\gamma(M)$ is direct sum of countably generated modules, whence summable, because γ is countable. Hereafter Theorem 2.2 is applicable to get that M is summable. If $\gamma = \alpha$, then F is separable summable, thus a direct sum of uniserial modules. Since γ is cofinal with ω, we obtain that α is cofinal with ω, hence $\alpha < \mu$ and then M is a direct sum of countably generated modules, whence summable. Q.E.D.

Remark 2.4. The preceeding theorem extends ([1, Theorem 2.11]) to summable modules. Moreover, in that Theorem 2.11, α = length of M should be cofinal with ω.

And so, we prepare to prove the following.

Theorem 2.5. If M is a λ-module with an α-large submodule L such that $\omega \leq \alpha \leq \lambda \leq \mu$, then M is summable precisely when L is summable.

Proof. " $\Rightarrow$ ". Foremost, if length of $M < \mu$, we consider two possibilities. Firstly, if length of $M = \lambda < \mu$, we get that M is a direct sum of countably generated modules. Hence, [1]'s result is applicable to deduce that L is totally projective of countable length, that is a direct sum of countably generated modules, whence summable. Otherwise, if length of $M < \lambda$, M is obviously a direct sum of countably generated modules and again the previous procedure can be applied.

After this, let us assume that length of $M = \mu$. We know by [4] that $H_\omega(L) = H_\eta(M)$ for some $\eta \le \alpha$. Hence $H_\omega(L)$ is summable. If $\eta < \alpha$, then L is a Σ-module. Therefore, L is summable. In the remaining case when $\eta = \alpha$, we have that $L/H_\omega(L) = L/H_\alpha(M)$ is an α-large submodule of the totally projective module $M/H_\alpha(M)$. Thus, in virtue of [4], $L/H_\omega(L)$ is a direct sum of uniserial modules. As above, L has to be a summable module.

" $\Leftarrow$ ". As already observed via [4], $H_\omega(L) = H_\eta(M)$. Consequently, L being summable secures that so is $H_\eta(M)$. Since $\eta \le \alpha \le \lambda$ and $\eta \le \mu$, we derive that $M/H_\eta(M)$ is a direct sum of countably generated modules, thus summable. Furthermore, we apply Theorem 2.2 to obtain the wanted claim. Q.E.D.

Proposition 2.6. Suppose M is a $QTAG$-module and γ is an ordinal.

(i) If M is summable, then $H_\gamma(M)$ and $M/H_{\gamma+1}(M)$ are summable.

(ii) If γ is countable and both $H_\gamma(M)$ and $M/H_\gamma(M)$ are summable, then M is summable.

Proof. (i) If $Soc(M)$ is the valuated direct sum of countably generated modules, then the same can clearly be said of $Soc(H_\gamma(M))$. Next, Let N be a $(\gamma+1)$-high submodule of M, and let $K = [N + H_{\gamma+1}(M)]/H_{\gamma+1}(M)$, so that $N \cong K$ are isotype in M and $M/H_{\gamma+1}(M)$, respectively. It follows that there are valuated direct sum decompositions

$$Soc(M) = Soc(N) \oplus P \quad \text{and} \quad Soc(M/H_{\gamma+1}(M)) = Soc(K) \oplus Q$$

where $H_{\gamma+1}(M) \subseteq P \subseteq H_\gamma(M)$ and $Q \subseteq H_\gamma(M/H_{\gamma+1}(M))$. Note that Q is, in fact, summable. Therefore, since M is summable, we can conclude $Soc(N)$ and P are summable. This, in tern, implies that $Soc(Q)$ and $H_{\gamma+1}(M)$ are summbale, so that $M/H_{\gamma+1}(M)$ and $H_{\gamma+1}(M)$ are summable. The fact that $H_{\gamma+1}$ is summable readily implies that $H_\gamma(M)$ shares this property.

(ii) Let N and P be as in part (i). Since $H_\gamma(M)$ is summable, we can infer that P has this property. We need, therefore, to show that $Soc(N)$ is also summable. Since $M/H_\gamma(M)$ is summable and γ is countable, we conclude that $Soc(M/H_\gamma(M))$ is the ascending union of submodules X_m for $m < \omega$ such that $\mathcal{S} = \{H_{M/H_\gamma(M)}(x) : x \in X_m\}$ is finite. If we let $Y_m = \{y \in N : y + H_\gamma(M) \in X_m\}$, then clearly $Soc(N)$ is the ascending union of the Y_m and $\{H_M(y) : y \in Y_m\} \subseteq \mathcal{S} \cup \{0\}$ is finite. Therefore, $Soc(N)$ is summable. Q.E.D.

Analysis. There are lots of examples of suumable modules M such that $M/H_\omega(M)$ is not summable; in fact suppose B is an unbounded direct sum of uniserial modules and $\overline{B}$ is its closure. Then $M = \overline{B}/Soc(B)$ can easily be seen to be summable, but on the other hand

$$M/H_\omega(M) = (\overline{B}/Soc(B))/(Soc(\overline{B})/Soc(B)) \cong \overline{B}/Soc(\overline{B}) \cong H_1(\overline{B})$$

is not summable. So Proposition 2.6(i) does not hold if $\gamma + 1$ is replaced by γ.

In addition, If M is a totally projective module of length γ, where $\omega_1 < \gamma < \omega_1.2$, then $H_{\omega_1}(M)$ and M/H_{ω_1} are both summable, but M is not; so Proposition 2.6(ii) does not hold for uncountable γ.

As an immediate consequence, we yield the following.

Corollary 2.7. Let M be a $QTAG$-module. Then M is summable if, and only if, $H_{\omega+1}(M)$ and $M/H_{\omega+1}(M)$ are summable.

The last statement can slightly be extended to the following.

Proposition 2.8. Let M be a *QTAG*-module. Then M is summable if, and only if, $H_{\omega+k}(M)$ and $M/H_{\omega+k}(M)$ are summable for some $k < \omega$.

Proof. First observe that

$$M/H_{\omega+1}(M) \cong (M/H_{\omega+k}(M))/(H_{\omega+1}(M)/H_{\omega+k}(M)) = (M/H_{\omega+k}(M))/H_{\omega+1}(M/H_{\omega+k}(M))$$

Moreover, since a module M is summable uniquely when so is $H_n(M)$ for some natural n, we derive that $H_{\omega+k}(M)$ is summable precisely when the same holds for $H_{\omega+1}(M)$. Henceforth, we apply the above corollary to infer the required claim. Q.E.D.

Lemma 2.9. If $N = Soc(H_{\omega+1}(M))$ and M/N is summable, then M is summable.

Proof. Note that

$$H_{\omega+2}(M) = H_1(H_{\omega+1}(M)) \cong H_{\omega+1}(M)/Soc(H_{\omega+1}(M)) = H_{\omega+1}(M)/N = H_{\omega+1}(M/N)$$

and, hence, $H_{\omega+2}(M)$ is summable. But this clearly implies that $H_{\omega+1}(M)$ is summable, and in view of Corollary 2.7 it remains only to show that $M/H_{\omega+1}(M)$ is also summable. Indeed, since $N \subseteq H_{\omega+1}(M)$ we have

$$M/H_{\omega+1}(M) \cong (M/N)/(H_{\omega+1}(M)/N) = (M/N)/H_{\omega+1}(M/N)$$

and the latter module is really summable because M/N is so by hypothesis. Q.E.D.

Corollary 2.10. If $N = M^\beta$ and M/N is summable, then $M/H_1(N)$ is summable.

Proof. Observe that

$$N/H_1(N) = Soc(H_\beta(M/H_1(N)))$$

where $\beta = \beta_0 = \omega + 1$. Therefore,

$$(M/H_1(N))/(N/H_1(N)) \cong M/N$$

is summable and hence by Lemma 2.9, $M/H_1(N)$ is summable. Q.E.D.

So, we come to the following sufficient condition for summability.

Theorem 2.11. Let M be a *QTAG*-module and $H_{\omega.2}(M) = 0$. If β is an increasing sequence of ordinals and symbols ∞ such that both M^β and M/M^β are summable, then M itself is summable.

Proof. Let $N = M^\beta$. If the sequence β contains any symbols ∞, then there is a positive integer n such that $H_n(N) = 0$. But then repeated applications of Corollary 2.10 yield the desired conclusion that $M/H_n(N) \cong M$ is summable. Thus we may assume that β is an increasing sequence of ordinals does not contain symbols of the type ∞, and take $\alpha = \sup(\beta)$ where $\beta = \{\beta_k\}_{k<\omega}$. It is not hard to see that $H_\omega(N) = H_\alpha(M)$ where $\alpha \leq \omega.2$.

Foremost, suppose for a moment $\alpha \leq \omega.2$, whence $H_\omega(N) = 0$ so that N is separable summable and thus a direct sum of uniserial modules. In view of [1], we obtain that M is σ-summable. Thus

$H_\omega(M)$ is σ-summable, i.e., a direct sum of uniserial modules and hence summable. Therefore $H_{\omega+k}(M)$ is summable for any natural number k.

Next, suppose that $\alpha < \omega.2$. Consequently, $H_\alpha(M)$ is summable. But $\alpha = \omega + m$ for some $m < \omega$, so that $H_{\omega+m}(M)$ being summable immediately forces that $H_{\omega+k}(M)$ is summable for all $k < \omega$.

Furthermore, in virtue of Proposition 2.8, we need only verify in the both cases for α alluded to above that $M/H_{\beta_k}(M)$ is summable for some $k \geq 1$, where $\beta_k = \omega+k$. But since $H_k(N) \subseteq H_{\beta_k}(M)$, we have again with Proposition 2.8 at hand that

$$M/H_{\beta_k}(M) \cong (M/H_k(N))/H_{\beta_k}(M/H_k(N))$$

is summable since each $M/H_k(N)$ is summable by the subsequent application k times of Corollary 2.10. Q.E.D.

3 Open problems

In conclusion, we pose two questions that we find interesting.

We recollect that an h-reduced $QTAG$-module M is said to be σ-summable (see [17]) if $Soc(M) = \cup_{n<\omega} M_n$, $M_n \subseteq M_{n+1}$ and for each $n < \omega$ there is an ordinal $\alpha_n <$ length of M such that $M_n \cap H_{\alpha_n}(M) = 0$. It is well-known that (see [1]) a σ-summable α-module of length α is totally projective; note that such an α is of necessity cofinal with ω and thus limit ordinal. By analogy, we ask the following.

Problem 3.1. Does it follow that a σ-summable module M of countable limit length for which $M/H_\beta(M)$ is summable for each non-limit $\beta <$ length of M is summable? If not, under what additional circumstances this is true?

Recall that a module M is a Σ-module (see [15]) if some its high submodule is a direct sum of uniserial modules. These modules are also known as layered modules in [7], that is, if $Soc(M) = \cup_{n<\omega} M_n$, where $M_n \subseteq M_{n+1} \subseteq Soc(M)$ and, for every $n \geq 1$, $M_n \cap H_n(M) = Soc(H_\omega(M))$. Clearly, every summable module is a Σ-module but the converse is not true in general. In fact a $QTAG$-module M is summable if and only if M is a Σ-module and $H_\omega(M)$ is summable. The role of the ordinal ω in this necessary and sufficient condition is crucial. Nevertheless, we ask the following.

Problem 3.2. Does it follow that a Σ-module M for which $H_\beta(M)$ is summable for some $\beta > \omega$ plus something else will imply that M is summable?

This is equivalent to find suitable conditions under which a Σ-module M whose submodule $H_\delta(M)$ is summable will ensures that $H_\eta(M)$ is summable whenever $\eta < \delta$.

Acknowledgements

The authors express their appreciation to the referee and to the editor, for his/her valuable editorial work.

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